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EXISTENCE AND UNIQUENESS FOR A CLASS OF CAUCHY-PROBLEMS WITH CHARACTERISTIC INITIAL MANIFOLDS

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0. Introduction

It is well known that the basic existence-uniqueness theorem in the theory of partial differential equations, the Cauchy-Kowalewski theorem, depends critically on the assumption that the initial data are given on a non-characteristic manifold. In general the situation may become quite irregular for characteristic initial value problems: the initial data cannot be prescribed arbitrarilly, the higher derivatives are not always determined by the equation. In (5) Leray describes solutions which are multivalued near initial manifolds which are characteristic along certain curves.

In this paper we give a satisfactory existence-uniqueness theorem for a large class of quasilinear second order equations with Cauchy data on a manifold which is characteristic in the strong sense that the equation is singular of order one along the initial manifold. The interest in this class of equations stems from the study of some basic elliptic equations, notably the minimal equation, in the presence of a Lie symmetry group. Reduction of the equation to the orbit space yields quasilinear elliptic equations, which have precisely such singularities along the boundary strata of the orbit space. From the point of view of global analysis, the most interesting closed solution hypersurfaces will be found among those which emanate from hypersurfaces of such singular boundary strata. In fact, in the case of cohomegeneity two Lie symmetry groups, where the reduced equation becomes an ordinary differential equation, there has been much recent progress (2), (3). The solution curves emanating from points on the boundary arcs have, for example, given significant new insights into the spherical Bernstein problem (3), (9), see also (1), where existence--uniqueness was discussed in the case of ordinary differential equations. In the case of higher cohomogeneity groups, with singular partial differential equations in orbit space, a different proof is required. We state our result in a sufficiently general form to cover the technical boundary problems for such reduced equations of constant mean curvature type. Applications and some concrete examples are discussed in the last section.

1. We fix some notation.

Let $x = (x_1,\dots,x_r)$, and consider a function $y = y(z,x_1,\dots,x_r)$, We use standard notation y_{x_i} for partial derivatives, and define $y_x = (y_{x_1},\dots,y_{x_r})$, $y_{zx} = (y_{zx_1},\dots,y_{zx_r})$, $y_{xx} = (y_{x_1x_1},\dots,y_{x_ix_j},\dots,y_{x_rx_r})$. We also use standard multi-index notation x^I for $x_1^{i_1}\dots x_r^{i_r}$, where $I = (i_1,\dots,i_r)$. Define $I \leqslant J = (j_1,\dots,j_r)$ if $i_k \leqslant j_k$, $k = 1,\dots,r$, $I < J$ if $I \leqslant J$ and $I \neq J$; and let $|I| = i_1+\dots+i_r$.

Theorem 1.
Consider the second order differential equation

$$(\ast) \quad zy_{zz} + \lambda y_z = zy_{zz}P(z,x,y,y_z,y_x) + y_zQ(z,x,y,y_z,y_x) + zF(z,x,y,y_z,y_x,y_{xx},y_{zx})$$

where P, Q, and F are analytic, P and Q without constant term, and $\lambda > 0$. Then:

(i) Any analytic solution $y = y(z,x)$ of $(\ast)$ defined in a neighbourhood of zero, satisfies the initial condition $y_z(0,x) = 0$.

(ii) For any analytic function of the form $f(x) = \Sigma a_I x^I$ with $a_I = 0$ for $|I| \leq 1$, there exists, locally around zero, a unique analytic solution $y(z,x)$ of $(*)$ which satisfies the initial conditions: $y(0,x) = f(x)$ and $y_z(0,x) = 0$.

Proof: The proof of (i) is trivial from the equation. Let $y(z,x) = \sum_{I,j} a_{I,j} z^j x^I$. Direct computation of the left hand side of $(*)$ gives $\sum (j+2)(j+\lambda+1) a_{I,j+2} z^{j+1} x^I$. The initial conditions give $a_{I,0} = a_I$, $a_{I,1} = 0$; and since $a_I = 0$ for $|I| \leq 1$ it follows that $y(z,x)$, $y_z(z,x)$, $y_{x_i}(z,x)$ have no constant terms. Let $P_1(z,x) = P(z,x,y(z,x),y_z(z,x),y_x(z,x))$, and define $Q_1(z,x)$ and $F_1(z,x)$ by corresponding substitutions in Q and F respectively. Then P_1 and Q_1 have no constant terms. We now collect terms of the type $z^{j+1} x^I$ on the right hand side of $(*)$, and get the following contributions: From zy_{zz}: $a_{I',j'+2} x^{I'} z^{j'+1}$ with either $j'<j$ or $j'=j$, $I'<I$. From $P_1(z,x)$ the entries which contribute to the $x^I z^{j+1}$-term are as follows: From $y(z,x)$: $a_{I',j'} x^{I'} z^{j'}$ with $j' \leq j$, $I' \leq I$, from $y_z(z,x)$: $a_{I',j'} x^{I'} z^{j'-1}$ with $j' \leq j+1$, $I' \leq I$, and from $y_{x_k}(z,x)$: $a_{i_1',\ldots,i_k'+1,\ldots i_r',j'} x^{I'} z^{j'}$ with $j' \leq j$ (in this case it is possible that the index $(i_1',\ldots,i_{k+1}',\ldots,i_r') > I$) Combining these, it follows that the coefficient of the $z^{j+1} x^I$-term in $zy_{zz} P_1(z,x)$ is a polynomial in the $a_{I',j'}$, $j' < j+2$ or $j' = j+2$ and $I' < I$, and the coefficients of P. The same argument works for $y_z\, Q_1(z,x)$. From the term $zF_1(z,x)$ we only need to check the contributions from y_{xx} and y_{zx}. From $y_{x_k x_j}$ there are terms of the type $a_{I'',j'} x^{I'} z^{j'}$ with $I'' = (i_1',\ldots,i_{k-1}',i_k'+1,i_{k+1}',\ldots,i_{j-1}', i_j'+1,i_{j+1}',\ldots,i_r')$ and $j' \leq j$, from y_{zx_k} of the type $a_{I'',j'+1} x^{I'} z^{j'}$ with $I'' = (i_1',\ldots,i'_k+1,\ldots,i_r')$, and $j' \leq j$. From the computation of the left hand side it now follows that we obtain a recursion formula which expresses $a_{I,j+2}$ as a polynomial in the $a_{I',j'}$ with $j' \leq j+1$ or $j' = j+2$, $I' < I$, and the coefficients of P, Q, F. Since

$a_{I,0}$ and $a_{I,1}$ are given by the initial conditions, the uniqueness part is now clear.

This recursion formula defines a formal power series for the solution. The standard majorization procedure to prove convergence does not work; however, a sufficiently careful choice of majorizing series still gives the desired convergence as follows: We first majorize the right hand side by substituting the power series $\bar{P}$, $\bar{Q}$, $\bar{F}$ obtained from the power series of P, Q, F by taking the absolute values of the coefficients. We then majorize the initial data by choosing $\bar{a}_{I,0} = |a_{I,0}|$, $\bar{a}_{I,1} = 0$ (notice that majorization by positive $\bar{a}_{I,1}$ would not work here). We now get the corresponding recursion formula for $\bar{a}_{I,j+2}$, hence $\bar{a}_{I,j+2} \geqslant |a_{I,j+2}|$, and it is sufficient to prove convergence of the formal power series for the majorized problem. Since $zy_{zz} = \sum(j+2)(j+1)\bar{a}_{I,j+2}\, x^I z^{j+1}$ and $y_z = \sum(j+2)\bar{a}_{I,j+2}\, x^I z^{j+1}$ we may again majorize the right hand side of the recursion formula for $\bar{a}_{I,j+2}$ by substituting $zy_{zz}\bar{Q}$ for $y_z\bar{Q}$. Since $\lambda>0$, we further majorize by dropping the term λy_z on the left hand side. We are now comparing with the recursion formula obtained from the equation $zy_{zz} = (\bar{P}+\bar{Q})zy_{zz} + z\bar{F}$; here z cancels, so we obtain $y_{zz} = (1-(\bar{P}+\bar{Q}))^{-1}\,\bar{F}(z,x,y,y_z,y_x,y_{xx},y_{zx})$. Here $\bar{P}$ and $\bar{Q}$ are without constant term, so the formal power series for the solution of this equation converges by the standard Cauchy-Kowalewski theorem.

q.e.d.

Remark: The result can obviously be generalized to other values of λ. For example, if $\lambda \in (0,-1)$, we can majorize by substituting λzy_{zz} for λy_z on the left hand side.

2. The restriction $a_I = 0$ for $|I| \leqslant 1$ on initial conditions in Theorem 1 is too restrictive for general applications. Under suitable translational and rotational invariance, however, the theorem holds for arbitrary analytic initial condition $y(0,x) = f(x)$.

We will consider the second order equation:

$$(**) \qquad zP(z,y_z,y_x,y_{zz},y_{zx},y_{xx}) + y_z\, P_4(z,y_z,y_x) + zF(z,x,y_z,y_x) = 0, \quad \text{where:}$$

(I) $P = P_1(z,y_z,y_x)y_{zz} + P_2(z,y_z,y_x)y_{zx} + P_3(z,y_z,y_x)y_{xx}$.

Here P_2 and P_3 take values in R^r and R^{r^2} respectively, and we take standard inner products with y_{zx} and y_{xx}.

(II) P is invariant under the Euclidean group in (x,y)-space (or a doubly transitive transformation group).

(III) P_1, P_2, P_3, P_4, F are analytic.

(IV) P_1 and P_4 are positive.

Theorem 2.

Consider the second order differential equation (**) with the above conditions (I)-(IV). Then:

(a) Any analytic solution of (**) defined in a neighbourhood of a point (o,x_0) in $R + R^r$ satisfies $y_z(0,x) = 0$

(b) For any analytic initial condition $f(x) = \sum a_I(x-x_0)^I$ there exists a unique analytic solutuon $y(z,x)$ of (**) such that $y(0,x) = f(x)$, $y_z(0,x) = 0$.

(c) This solution depends analytically on any s-parameter variation of f.

Proof: Since the form of (**) and (I)-(IV) are preserved under translations in (y,x)-space, we may assume $x_0 = 0$ and $y(0,0) = a_{0,0} = 0$. The tangent space of the graph of $y(0,x) = f(x) = \sum a_I x^I$ is given by the hyperplane $y = \sum_{|I|=1} a_I x^I$, by a rotation in (x,y)-space we obtain the new equation $y' = 0$ for this, hence the initial condition (ii) of Theorem 1 now holds. We have:

$y = A_{00}y' + \sum A_{0i}x'_i$

$x_j = A_{j0}y' + \sum A_{ji}x'_i$

and let B be the inverse matrix of (A_{ij}).

Then a computation gives $y_{x_k} = A_{00} \sum_j y'_{x_j}(x'_j)_{x_k} + \sum_j A_{0j}(x'_j)_{x_k}$.

Substitution of $(x'_j)_{x_k} = B_{jk} + B_{j0}\, y_{x_k}$ and solving gives

$$y_{x_k} = (B_{00} - \sum B_{j0}y'_{x_j})^{-1}(\sum B_{jk}y'_{x'_j} - B_{0k}).$$

Similarly, we compute $y_z = (B_{00} - \sum B_{j0}y'_{x'_j})^{-1}y'_z$.

Since the y-axis is not contained in the tangent space of $y = \sum a_I x^I$, it follows that $B_{00} \neq 0$, and the y_{x_k}, y_z are power series in the y'_z, $y'_{x'_j}$.

By hypothesis $zP(z, y_z, y_x, y_{zz}, y_{zx}, y_{xx})$
$= zP(z, y'_z, y'_{x'}, y'_{zz}, y'_{zx'}, y'_{x'x'})$, furthermore $P'_4(z, y'_z, y'_{x'}) = P_4(z, y_z, y_x)$ is positive by hypothesis, hence it has a positive constant term λ. In the new coordinate system (z, x', y') the transformed equation with the given initial contitions now satisfies all conditions of Theorem 1. This proves (a) and (b) of Theorem 2. The recursion formula for the coefficients of the solution will express $a_{I,j+2}$ as a polynomial in the a_I's; hence (c) follows.

q.e.d.

3. In this section we demonstrate that elliptic partial differential equations with singularities of the type considered in Theorems 1, 2, occur naturally as reduced miminal (or constant mean curvature) equations on orbit spaces, and hence constitute an important object of study.

Let M be a Riemannian manifold with a compact group of isometries G. A hypersurface N of constant mean curvature is characterized (locally) by a non-linear partial differential equation of elliptic type. Let $\pi. M \to X = M/G$ be the orbit projection and let $\pi|M^*: M^* \to X^*$ be the restriction of π to the open, dense subset M^* of points on principal orbits; then $\pi|M^*$ is a Riemannian submersion when X is equipped with the orbital distance metric (O'Neill (6)). If a hypersurface N is G-invariant, there is a simple formula for its mean curvature in terms of the geometry of the orbit space and the fibers, leading to a reduction of the above equation to a non-linear differential equation with singularities in orbit space.

Proposition 3.

Let M, G, X, π, M^* be as above, and let N be a G-variant hypersurface of M with $N^* = \pi(N) \cap X^*$. Let p be on a principal orbit P in N, let e be a unit normal vector to P at p, and $n = \pi_*(e)$. Then $H(e) = H'(n) - \frac{d}{dn} \ln v$, where $H(e)$ and $H'(n)$ are the mean curvatures of N and N^* in the directions of e and n reespectively, the volume function v on X is defined by $v(x) = \mathrm{vol}(\pi^{-1}(x))$, and $\frac{d}{dn}$ is the directional derivative along the normal n.

Remark. This result from equivariant geometry has been applied in several recent papers (2), (9). It can be demonstrated by applying the first variational formula for the volume of N to compactly supported, equivariant variations.

The orbit invariants are (even in the case of linear representations) computable only in special cases. Prominent among those are (A) the isotropy representation of a compact symmetric space $M = H/K$, and (the closely related) (B) isotropy action of K on H/K by left translations. Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of H, and let $\mathfrak{a}$ be maximal Abelian in $\mathfrak{p}$.

Let Σ^+ be a system of positive restricted roots on $\mathfrak{a}$. Then the restricted Weyl group W (respectively the affine Weyl group Γ) is generated by reflections in the hyperplanes annihilated by restricted roots (respectively where the values of the restricted roots are integer multiples of πi). The next proposition follows from Cartan-Weyl theory.

Proposition 4.

(A) For the isotropy representation of H/K: The orbit space $\mathfrak{p}/K \simeq \mathfrak{a}/W$; i.e. the orbit space can be identified with a flat Weyl chamber C in $\mathfrak{a}$. For an interior point x of C, the volume of the principal orbit $K\cdot x$ is given by $v(x) = \prod_{\alpha\in\Sigma^+} |i\alpha(x)|$.

(B) For the isotropy action: The orbit space $K\backslash H/K \simeq \mathfrak{a}/\Gamma$, i.e. the orbit space can be identified with a flat Cartan polyhedron D in $\mathfrak{a}$. For an interior point x of D, the volume of the principal orbit KxK is given by $v(x) = \prod_{\alpha\in\Sigma^+} |\sin i\alpha(x)|$.

In this type of situation we can then write $v = w_1^{p_1}\cdots w_s^{p_s}$, $w_j = h(\alpha_j)$, where α_j is a linear functional, and $h(\alpha_j) = i\alpha_j$ in case (A), $h(\alpha_j) = \sin i\alpha_j$ in case (B). Then $\frac{d}{dn}\ln v = \sum_j \frac{p_j}{w_j}\frac{d}{dn}w_j$, and from Proposition 3 it follows that the reduced constant mean curvature equation in orbit space, is

$$(***)\qquad H' - \sum_{j=1}^{s} \frac{p_j}{w_j}\frac{d}{dn} w_j = C.$$

This equation is singular along the hyperplanes $w_j = 0$. Let p be a generic point on such a boundary stratum, say $w_1(p) = 0$. There are two possibilities:

(a) Case (A) or case(B) when $2\alpha_1$ is not a restricted root, Then $w_j(p) \neq 0$ for $j > 1$, and the singular term of (***) is $\frac{p_1}{w_1}\frac{d}{dn}w_1$.

(b) Case (B) when $2\alpha_1$ is a restricted root; we may choose $\alpha_2 = 2\alpha_1$. Then $w_j(p) \neq 0$ for $j > 2$, and the singular term of (***) is $\frac{p_1}{w_1}\frac{d}{dn} w_1 + \frac{p_2}{w_2}\frac{d}{dn} w_2$. Now choose Euclidean coordinates $(x_0, x_1, \ldots, x_{n+1})$ in X centered at p such that $i\alpha_1 = x_0$. For a hypersurface N^* of D through p given by the graph $x_{n+1} = f(x_0, \ldots, x_n)$ the mean curvature is given by $H' = (1+\|\nabla f\|^2)^{-3/2}$ $\{\sum_{i=0}^{n}(1+f_{x_0}^2+\ldots+f_{x_{i-1}}^2+f_{x_{i+1}}^2+\ldots+f_{x_n}^2)f_{x_i x_i} - 2\sum_{i<j} f_{x_i} f_{x_j} f_{x_i x_j}\}$.

The unit normal $n = (1+\|\nabla f\|^2)^{-1/2}\{-\sum_{i=0}^{n} f_{x_i}\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{n+1}}\}$.

Since $\frac{d}{dn}\ln v$ has no second order term, it follows that (***) is absolutely elliptic. To comply with the notation of Theorem 2 we now define $z = x_0$, $y = x_{n+1}$. Multiplying by $z(1+\|\nabla f\|^2)^{3/2}$, we can now write the reduced constant mean curvature equation (***) as follows:

$$z\{(1+y_{x_1}^2+\ldots+y_{x_n}^2)y_{zz}+\sum_{i=1}^{n}(1+y_z^2+y_{x_1}^2+\ldots+y_{x_{i-1}}^2+y_{x_{i+1}}^2+\ldots+y_{x_n}^2)y_{x_i x_i}$$

$$-2\sum_{i=1}^{n} y_z y_{x_i} y_{zx_i} - 2\sum_{1\leq i<j\leq n} y_{x_i} y_{x_j} y_{x_i x_j}\} + (1+y_z^2+y_{x_1}^2+\ldots+y_{x_n}^2)$$

$$\{p_1 \frac{z}{w_1} y_z h_1'(z)+z\sum_{j=2}^{s}[\frac{p_j}{w_j}(y_z (w_j)_z+\sum_{i=1}^{n} y_{x_i}(w_j)_{x_i}-(w_j)_y]$$

$$-zC(1+y_z^2+y_{x_1}^2+\ldots+y_{x_n}^2)^{3/2} = 0, \qquad \text{where } C \text{ is a constant.}$$

In case (a) we have $w_2(p) \neq 0$, and $P_4(z, y_z, y_{x'}) =$ $= (1+y_z^2+y_{x_1}^2+\ldots+y_{x_n}^2)p_1 \frac{z}{w_1} h_1'(z)$, where $\frac{z}{w_1} h_1'(z) = 1$ in case (A) and $\frac{z}{w_1} h_1'(z) = z \cot z$ in case (B); hence P_4 is analytic and positive in a neighbourhood of $z = 0$. Similarly, in case (b) we have $w_2(p) = \sin 2i\alpha_1(p) = 0$, and

$P_4 = (1+y_z^2+y_{x_1}^2+\ldots+y_{x_n}^2)(p_1 z \cot z+2p_2 z \cot 2z)$, which is again analytic and positive near $z = 0$. Also, $P_1 = (1+y_z^2+y_{x_1}^2+\ldots+y_{x_n}^2)$ is positive, and (I) and (IV) of Theorem 2 are satisfied. The terms of $F(z,x,y_z,y_x)$ of the form $\frac{p_j}{w_j}(y_z(w_j)_z+\sum y_{x_i}(w_j)_{x_i}-(w_j)_y)$ have $w_j(p) \neq 0$, hence they are analytic near $z = 0$, and (III) is satisfied. It follows from the symmetry properties of Euclidean geometry that the mean curvature H' is rotationally invariant, hence the final condition (II) of Theorem 2 is also satisfied. We have proved

Theorem 5.

Let X be either (A) a restricted Weyl chamber, or (B) a Cartan polyhedron of a compact, symmetric space H/K. Let p be an interior point on one of the walls H_j of X. Then, for any analytic hypersurface S of H_j through p, there is a unique, analytic solution $\bar{N}$ of the reduced constant mean curvature equation on X, defined in a neighbourhood of p, intersecting H_j along S, and depending analytically on S. The preimage of $\bar{N}$ in (A) the tangent space of H/K (at K) or (B) in H/K, is a constant mean curvature hypersurface without singularities.

Example 1. Let $SO(m)$ act on $R^m \oplus R^2$ by the standard representation on R^m and trivially on R^2. The orbit space is the upper half-space $z \geq 0$ of R^3, the volume function $f(x,y,z) = z^{m-1}$, and the resulting reduced constant mean curvature equation is:

$$z(1+y_x^2)y_{zz}- 2zy_xy_zy_{xz}+ z(1+y_z^2)y_{xx}+ 2(m-1)y_z(1+y_x^2+y_z^2)$$
$$= 2z(1+y_x^2+y_z^2)^{3/2}.$$

From any analytic curve C in the xy-plane, there emanates a unique, analytic, local solution surface, whose inverse image is a smooth constant mean curvature hypersurface of R^{m+2}.
A solution surface with an analytic Jordan curve C in the xy-plane as boundary defines a closed hypersurface of constant mean curvature (soap bubble) in R^{m+2}.

Example 2. Let $\phi = \rho\oplus 1$ on $R^{2m}\oplus R$, where ρ is the isotropy representation of the Grassmannian $SO(m+2)/SO(2)\times SO(m)$ of oriented

2-planes in R^{m+2}. The restricted root system in the Cartan subalgebra $\mathfrak{n}$, parametrized as the xz-plane, is given by $\pm i(x-z)$, $\pm i(x+z)$ with multiplicity 1, and $\pm ix$, $\pm iz$ with multiplicity $(m-2)$. The Weyl chamber is given by $z \geqslant 0$, $x \geqslant z$, and the volume formula by $v = x^{m-2}z^{m-2}(x-z)(x+z)$. The orbit space of R^{2m+1} is given by the cylinder $z \geqslant 0$, $x \geqslant z$ in (x,z,y)-space, and the reduced minimal equation is

$$(1+y_x^2)y_{zz} - 2y_x y_z y_{xz} + (1+y_z^2)y_{xx} + (1+y_x^2+y_z^2)\left[(m-2)\frac{y_x}{x} + (m-2)\frac{y_z}{z} + \frac{y_x-y_z}{x-z} + \frac{y_x+y_z}{x+z}\right] = 0$$ in local coordinates.

It follows from Theorem 5 that for each point p in the interior of one of the walls $z = 0$ or $z = x$, and for each analytic curve C through p, there exists locally a unique analytic solution through C, whose lift to R^{2m+1} is a smooth, minimal hypersurface.

Remark.

The above approach can easily be adapted to cover situations of the following type: Consider the representation $\phi \oplus 1$ on R^{2m+2}, where ϕ is as in Ex. 2, and restrict the action to the unit sphere S^{2m+1}. The orbit space is now a spherical domain of S^3 defined by $z \geqslant 0$, $x \geqslant z$. Introducing spherical co rdinates in $S^3 \subseteq R^4 = \{x,z,y,u)\}$ by $u = \cos\theta_3$, $y = \sin\theta_3 \cos\theta_2$, $z = \sin\theta_3 \sin\theta_2 \cos\theta_1$, $x = \sin\theta_3 \sin\theta_2 \sin\theta_1$, it is clear that the mean curvature term in $(\theta_1,\theta_2,\theta_3)$-space is again invariant under rotations of S^3; and Theorem 2 may be reformulated to cover this case. It would be interesting to know if suitable analytic curves on the singular boundary generate non-equatorial minimally imbedded hyperspheres in S^{2m+1} in this case (for $m \geqslant 4$ such examples are still unknown).

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